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DESIGN OF THE CYLINDRICAL SHELL FOR CYCLIC LOADING

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ABSTRACT. The asymptotic integration method, previously used for the design of shells of revolution for cyclic loading, is used to obtain a solution for the closed cyclic shell of random cross section subjected to cyclic loading.

Previously [1] the asymptotic integration method was used for the design of shells of revolution for cyclic loading. In this article a solution of equations by means of this method was obtained for the closed cyclic shell of random cross section subjected to cyclic loading. /106*

1. Let us take an equation of the inner surface of the cylindrical shell in the form

$$r = \xi i + yi + zk \tag{1.1}$$

Here ξ is the coordinate along the generatrix, $y = y\ (\eta)$ and $z = z\ (\eta)$ are the equations for the cross section of the cylinder where η is the coordinate along this line, where the first quadratic form of the cylinder is $ds^2 = d\xi^2 + d\eta^2$.

Since $\xi = c$ and $\eta = c$ are the lines of curvature along the inner surface of the shell, the principal radii of curvature are

$$\frac{1}{R_1} = 0, \qquad \frac{1}{R_2} = -\frac{y'z'' - z'y''}{(y'^2 + z'^2)^{3/2}} = \frac{1}{\rho}, \qquad \rho = \rho(\eta) \tag{1.2}$$

Let us introduce the dimensionless coordinates α and β

$$\xi = \lambda\alpha, \quad \eta = \lambda\beta \tag{1.3}$$

Here λ is some characteristic radius of curvature of the cross section of the cylinder. In this case

$$ds^2 = \lambda^2\ (d\alpha^2 + d\beta^2) \tag{1.4}$$

and ρ is a function of β.

As a starting equation let us take [2]

$$L\ (W) + h^{\circ}N\ (W) = 0 \tag{1.5}$$

*Numbers in the margin indicate pagination in the foreign text.

where the differential operators L and N for the cylindrical shell acquire the form

$$L = \frac{i}{\lambda^2}\frac{1}{\rho}\frac{\partial^2}{\partial\alpha^2}, \qquad N = \frac{i}{\lambda^4}\left(\frac{\partial^4}{\partial\alpha^4} + 2\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + \frac{\partial^4}{\partial\beta^4}\right) \tag{1.6}$$

Equation (1.5) is used for the design of a cylindrical shell making the following assumptions:

1) The shell must not be extremely long.

2) For each cross section function ρ should not have significant relative deviations from some mean value of ρ_0.

3) The shape of the cross section should not have areas with sharply changing curvature, so that during differentiation with respect to β the quantity ρ would not change significantly.

Let us consider four groups of boundary conditions when the cross sections of the cylinder $\alpha = \alpha_j$ (j = 1.2), displacements u_1, v_1, w_1 and the angle of rotation γ_1 will have the values

$$\left.\begin{array}{llll} u_1 = \cos k\beta, & v_1 = 0, & w_1 = 0, & \gamma_1 = 0 \\ u_1 = 0, & v_1 = \sin k\beta, & w_1 = 0, & \gamma_1 = 0 \\ u_1 = 0, & v_1 = 0, & w_1 = \cos k\beta, & \gamma_1 = 0 \\ u_1 = 0, & v_1 = 0, & w_1 = 0, & \gamma_1 = \cos k\beta \end{array}\right| \tag{1.7}$$

assuming that k is a sufficiently large number. Solutions may be obtained by superimposition when at the boundaries $\alpha = \alpha_j$ (j = 1.2) the quantities u_1, v_1, w_1 and γ_1 are represented in the Fourier series, which do not contain terms with small values of k.

The solution of Eq. (1.5) is a function of independent variables α, β and parameters h° and k. The latter are related by

$$k = (h^\circ)^{-t}, \qquad t = \frac{1}{2+\gamma} \tag{1.8}$$

Here exponent t is index of the change of the permissible load. The numbers γ and t are rational numbers, which, as shown in [3] and [1] does not decrease their generality.

We shall seek the stressed state of the cylinder, when γ is a positive number or according to (1.8), t < 1/2. In this case, as shown in [3] and [1], the solution of equation (1.5) has the form

$$W = W^{(1)} e^{kf} + W^{(2)} e^{\mu g} \tag{1.9}$$

where the first term is the integral corresponding to the principal stressed state of the shell, the second term is the integral corresponding to a simple edge effect, k is a parameter which enters the boundary conditions of Eq. (1.7), and

$$\mu = (h^\circ)^{-1/2} \tag{1.10}$$

The variability functions f and g have the form /107

$$\varphi = \varphi_0 + k^{-\varkappa/\xi}\varphi_1 + k^{-\varkappa+1/\xi}\varphi_2 + \cdots + k^{-(\xi-1)/\xi}\varphi_{\xi-\varkappa} \qquad (1.11)$$

Here φ is any of the two functions: f or g, and $W^{(1)}$ and $W^{(2)}$ are expressed as follows

$$W^{(j)} = W_0^{(j)} + k^{-1/\xi}W_1^{(j)} + \cdots + k^{-(r-1)/\xi}W_{r-1}^{(j)} + k^{-r/\xi}W_r^{(j)} \quad (j = 1, 2) \qquad (1.12)$$

For the principal stressed state and a simple edge effect we shall select $\varkappa/\xi$ in such a manner that a recurrent system of differential equations is obtained (independent of parameter k) with respect to all functions which enter (1.10), with the exception of the residual terms $W_r^{(j)}$ (j = 1.2).

It is known from [3] that when t < 1/2 the form of the integral of the principal stressed state is a function of the sign of the curvature of the shell. Since for the cylinders the curvature is equal to zero, operator L in (1.6) will be parabolic. Its characteristics (double) are determined by the equation $\beta = c$ and coincide with the rectilinear generatrices of the cylindrical shell. This means, as we shall see later, that in certain cases the effect of the edge loading is propagated into the interior of the shell along the asymptotic lines of the cylinder. It follows from here that the principal stressed state of the cylinder is not always attenuated as one moves further away from the edge along the generatrix. Thus, in certain cases, for the cylinder the San Venon principle is not as clearly pronounced as for shells with positive curvature. In constructing integrals, corresponding to the principal stressed state, one may assume that for the variability function, f

$$\varkappa/\xi = 1/2\gamma \qquad (1.13)$$

Then, taking $\gamma = p/q$ (p, q are simple whole numbers), one may consider that in (1.11) and (1.12) we have $\xi = 2q$. This means that

$$f = f_0 + k^{-p/2q}f_1 + \cdots + k^{-(2q-1)/2q}f_{2q-p} \qquad (1.14)$$

In order to fulfill the conditions of (1.7) it is necessary for the function f to have at the boundaries $\alpha = \alpha_1$ and $\alpha = \alpha_2$ the following values

$$f = \pm i\beta \qquad (1.15)$$

From this it stems that when $\alpha = \alpha_1$ and $\alpha = \alpha_2$

$$f_0 = \pm i\beta, \quad f_1 = 0, \ldots, \quad f_{2q-p} = 0 \qquad (1.16)$$

The variability function for the integrals, corresponding to a simple edge effect, is constructed just as it was done in [1]. According to this we obtain:

$$\mu g = k^{1+1/2\gamma}(g_0 + k^{-1/2\gamma}g_1 + k^{-1/2\gamma-1/2q}g_2 + \cdots + k^{-1/2\gamma-(2q-1)/2q}g_{\gamma q+2q}) \qquad (1.17)$$

where at the boundaries $\alpha = \alpha_1$ and $\alpha = \alpha_2$ we have

$$g_0 = 0, \quad g_1 = \pm i\beta, \quad g_l = 0 \quad (l = 2, \ldots, \gamma q + 2q) \qquad (1.18)$$

The nature of the stressed state of the cylindrical shell will depend on the value of γ. Subsequently we shall show that if $0 < \gamma < 2$, then for a short cylinder the stressed state is attenuated in the interior of the shell. However, when $\gamma \geqslant 2$ attentuation for the short shell does not occur.

For a cylindrical shell Eq. (1.5) with boundary conditions (1.7) will be solved for specific values of γ, namely when $\gamma = 2/3,\ 1,\ 2$. Here, basically different principal stressed states are developed in the shell.

2. Let us consider a case when $\gamma = 2/3$. From (1.14) and (1.17) we obtain the variability function

$$f = f_0 + k^{-1/3} f_1 + k^{-2/3} f_2, \quad g = g_0 + k^{-1/3} g_1 + k^{-2/3} g_2 + k^{-1} g_3 \tag{2.1}$$

The boundary conditions for the function f are given by the formulae (1.16), and for g by the formulae (1.18). In the second expansion of (1.12) for $W^{(j)}$ we must assume that $\xi = 3$.

Let us determine f, g and $W_0^{(1)}$. For this purpose we shall substitute the principal stressed state integral from (1.9) into (1.5), taking into account (1.6) and reducing by $k^2 e^{kf}$, we obtain

$$\begin{aligned} &L_0 W_0^{(1)} + k^{-1/3} \sum_{\sigma=0}^{1} L_{1/3\sigma} W_{1/3(1-\sigma)} + k^{-2/3} \left(\sum_{\sigma=0}^{2} L_{1/3\sigma} W_{1/3(2-\sigma)}^{(1)} + N_0 W_0^{(1)} \right) + \\ &+ k^{-1} \left(\sum_{\sigma=0}^{3} L_{1/3\sigma} W_{1/3(3-\sigma)} + \sum_{\sigma=0}^{1} N_{1/3\sigma} W_{1/3(t-\sigma)}^{(1)} \right) + k^{-4/3} \left(\sum_{\sigma=0}^{4} L_{1/3\sigma} W_{1/3(4-\sigma)}^{(1)} + \right. \\ &\left. + \sum_{\sigma=0}^{\sigma} N_{1/3\sigma} W_{1/3(2-\sigma)} \right) + \cdots \end{aligned} \tag{2.2}$$

The operators $L_{1/3}\sigma$ have the following form: /108

$$\begin{aligned} &L_0 = \frac{i}{\lambda^3 \rho} \left(\frac{\partial f_0}{\partial \alpha} \right)^2, \quad L_{1/3} = \frac{2i}{\lambda^3 \rho} \frac{\partial f_0}{\partial \alpha} \frac{\partial f_1}{\partial \alpha}, \quad L_{2/3} = \frac{i}{\lambda^3 \rho} \left[\left(\frac{\partial f_1}{\partial \alpha} \right)^2 + 2 \frac{\partial f_0}{\partial \alpha} \frac{\partial f_2}{\partial \alpha} \right] \\ &L_1 = \frac{i}{\lambda^3 \rho} \left(2 \frac{\partial f_0}{\partial \alpha} \frac{\partial}{\partial \alpha} + 2 \frac{\partial f_1}{\partial \alpha} \frac{\partial f_0}{\partial \alpha} + \frac{\partial^2 f_0}{\partial \alpha^2} \right), \quad L_{4/3} = \frac{1}{\lambda^3 \rho} \left[2 \frac{\partial f_1}{\partial \alpha} \frac{\partial}{\partial \alpha} + \frac{\partial^2 f_1}{\partial \alpha^2} + \left(\frac{\partial f_2}{\partial \alpha} \right)^2 \right] \\ &L_{5/3} = \frac{i}{\lambda^3 \rho} \left(2 \frac{\partial f_2}{\partial \alpha} \frac{\partial}{\partial \alpha} + \frac{\partial^2 f_2}{\partial \alpha^2} \right), \quad L_2 = \frac{i}{\lambda^3 \rho} \frac{\partial^2}{\partial \alpha^2} \end{aligned} \tag{2.3}$$

and N_0, $N_{1/3}$, and $N_{2/3}$ the form

$$\begin{aligned} &N_0 = \frac{1}{\lambda^4} \left[\left(\frac{\partial f_0}{\partial \alpha} \right)^2 + \left(\frac{\partial f_0}{\partial \beta} \right)^2 \right] \\ &N_{1/3} = \frac{4}{\lambda^4} \left[\left(\frac{\partial f_0}{\partial \alpha} \right)^3 \frac{\partial f_1}{\partial \alpha} + \frac{\partial f_0}{\partial \alpha} \frac{\partial f_1}{\partial \alpha} \left(\frac{\partial f_0}{\partial \beta} \right)^2 + \left(\frac{\partial f_0}{\partial \alpha} \right)^2 \frac{\partial f_0}{\partial \beta} \frac{\partial f_1}{\partial \beta} + \left(\frac{\partial f_0}{\partial \beta} \right)^3 \frac{\partial f_1}{\partial \beta} \right] \\ &N_{2/3} = \frac{1}{\lambda^4} \left\{ 4 \left(\frac{\partial f_0}{\partial \alpha} \right)^3 \frac{\partial f_2}{\partial \alpha} + 6 \left(\frac{\partial f_0}{\partial \alpha} \right)^2 \left(\frac{\partial f_1}{\partial \alpha} \right)^2 + 2 \left(\frac{\partial f_0}{\partial \alpha} \right)^2 \left[2 \frac{\partial f_0}{\partial \beta} \frac{\partial f_2}{\partial \beta} + \left(\frac{\partial f_1}{\partial \beta} \right)^2 \right] + \right. \\ &\left. + 8 \frac{\partial f_0}{\partial \alpha} \frac{\partial f_1}{\partial \alpha} \frac{\partial f_0}{\partial \beta} \frac{\partial f_1}{\partial \beta} + 2 \left(\frac{\partial f_0}{\partial \alpha} \right)^2 \left[2 \frac{\partial f_0}{\partial \alpha} \frac{\partial f_2}{\partial \alpha} + \left(\frac{\partial f_1}{\partial \alpha} \right)^2 \right] + 6 \left(\frac{\partial f_0}{\partial \beta} \right)^2 \left(\frac{\partial f_1}{\partial \beta} \right)^2 + 4 \left(\frac{\partial f_0}{\partial \beta} \right)^3 \frac{\partial f_2}{\partial \beta} \right\}, \ldots \end{aligned} \tag{2.4}$$

By equating coefficients of $k^{-1/3\sigma}$ from (2.2) to zero and keeping in mind (2.3) and (2.4) we obtain the following differential equations with respect to f_0, f_1, f_2 and $W_0^{(1)}$:

$$L_0 W_0^{(1)} = 0, \quad L_{1/3} W_0^{(1)} = 0, \qquad (L_{2/3} + N_0) W_0^{(1)} = 0$$
$$(L_1 + N_{1/3}) W_0^{(1)} = 0, \qquad (L_{4/3} + N_{2/3}) W_0^{(1)} = 0 \tag{2.5}$$

Substituting operators L_0 and $L_{1/3}$ into the first two equations of (2.6), we find the equations for f_0 and f_1

$$\left(\frac{\partial f_0}{\partial \alpha}\right)^2 = 0, \qquad \frac{\partial f_0}{\partial \alpha}\frac{\partial f_1}{\partial \alpha} = 0 \tag{2.6}$$

Both of these equations and boundary conditions (1.6) are fulfilled when $f_0 = \pm i\beta$.

This occurs because $\beta = c$ are double characteristics of L. For the calculation of f_1 we make use of a third equation in (2.5). In addition to operator $L_{2/3}$ it includes N_0. This means that function f_1 depends not only on the momentless, but also on the moment containing operator N. Keeping in mind the boundary condition (1.16) and formula $f_0 = \pm i\beta$, we find

$$f_1 = \pm \frac{1+i}{\sqrt{2}}\left(\frac{\rho}{\lambda}\right)^{1/2}(\alpha - \alpha_j) \tag{2.7}$$

From the fourth equation of (2.5) we obtain the function

$$f_2 = \pm {}^1\!/_2 (\rho'/\lambda) \quad (\alpha - \alpha_j)^2 \tag{2.8}$$

Thus, in the case of boundary condition (1.16) the function f acquires the form

$$kf = \pm ik\beta \pm k^{2/3}\frac{1+i}{\sqrt{2}}\left(\frac{\rho}{\lambda}\right)^{1/2}(\alpha - \alpha_j) \pm k^{1/3}\frac{\rho'}{2\lambda}(\alpha - \alpha_j)^2 \tag{2.9}$$

The first term in the right side of (2.9) gives the solution, which oscillates without attenuation. The sign of the second term is selected to be such that in the zone, adjacent to the considered edge, the real part would be negative, i.e. in order for attenuation to take place into the interior.

From the fourth equation (2.5), using (2.9) we find a differential equation for the function $W_0^{(1)}$

$$\frac{\partial W_0^{(1)}}{\partial \alpha} \mp \frac{1-i}{\sqrt{2}}\left(\frac{\rho}{\lambda}\right)^{1/2}\left[1 + 2\frac{\rho'}{\rho}(\alpha - \alpha_j) + \frac{\rho'^2}{4\rho^2}(\alpha - \alpha_j)^2\right] W_0^{(1)} = 0 \tag{2.10}$$

The solution of this equation shall be /109

$$W_0^{(1)} = (a + ib)\, e^{\psi}, \quad \psi = \pm \frac{1-i}{\sqrt{2}}\left(\frac{\rho}{\lambda}\right)^{1/2}(\alpha - \alpha_j)\left[1 + \frac{\rho'}{\rho}(\alpha - \alpha_j) + \frac{\rho'^2}{12\rho^2}(\alpha - \alpha_j)^2\right] \tag{2.11}$$

Here a and b are random functions of β, determined by (1.17). The equations for $W_{1/3}^{(1)}$ and $W_{2/3}^{(2)}$ have the same form as (2.10), but they are now homogeneous.

Thus, the integral of the principal stressed state (1.9) has the form

$$W^{(1)} e^{kf+\psi}$$

In order for the stressed state to be attenuated towards the interiof of the cylinder, it is necessary for a real part of the exponent kf + Ψ to be negative. For this purpose it is necessary to limit the length of the shell $l = \lambda\ (\alpha_2 - \alpha_1)$ by a strong inequality

$$l \ll k^{1/3} \left| \frac{\lambda \sqrt{2\lambda\rho}}{\rho'} \right| \tag{2.12}$$

(long shells were excluded from consideration at the beginning of this article).

Let us now determine the variability functions of the edge effect. By substituting the second term in Eq. (1.9) in the differential Eq. (1.5) and taking into account (2.1) and (1.17), we obtain the expression

$$\sum_{\sigma=0}^{r} \mu^{-\frac{\sigma}{4}} \sum_{j=0}^{\sigma} (L_j + N_j) W_{\sigma-j}^{(2)} = 0 \tag{2.13}$$

The operators L_j, N_j are determined by formulas analogous to (2.3) and (2.4) in which it is only necessary to replace f by g.

By setting coefficients in front of equal powers of μ to zero we obtain the differential equation for the sought functions g_0, g_1, g_2, g_3. Thus, the equation for g_0 will have the form

$$\left[\left(\frac{\partial g_0}{\partial \alpha}\right)^2 + \left(\frac{\partial g_0}{\partial \beta}\right)^2\right]^2 + i\,\frac{\lambda}{\rho}\left(\frac{\partial g_0}{\partial \alpha}\right)^2 = 0 \tag{2.14}$$

We shall solve this equation taking into account the first condition in (1.18) and impose the requirement on the integral corresponding to a simple edge effect, to be attenuated from the edge into the interior of the shell. Taking this into account we seek the solution for (2.14) in the form of $\alpha - \alpha_j$ powers series

$$g_0 = \frac{\alpha - \alpha_j}{1!}\, g_0^{(0)} + \frac{(\alpha - \alpha_j)^2}{2!}\, g_0^{(1)} + \cdots \tag{2.15}$$

Here $g_0^{(0)}$, $g_0^{(1)}$ are derivatives of g with respect to α when $\alpha = \alpha_j$ and are only the functions of β.

Since when $\alpha = \alpha_j$, $g_0 = 0$, then at the boundary $\alpha = \alpha_j$ all of the derivatives of g_0 with respect to $\beta(\partial g_0/\partial\beta,\ \partial^2 g_0/\partial\beta^2, \ldots)$ are equal to zero. Keeping this in mind and using (2.14) we determine $g_0^{(1)}$. Differentiating (2.14) with respect to α and substituting $\alpha = \alpha_j$, we obtain $g_0^{(2)}$, etc. As a result of the solution of (2.15) for g_0 it becomes

$$g_0 = \pm \frac{1-i}{\sqrt{2}} \left(\frac{\rho}{\lambda}\right)^{1/2} (\alpha - \alpha_j) + \frac{\partial^3 g_0}{\partial \alpha^3} \frac{(\alpha - \alpha_j)^3}{3!} + \cdots$$

Functions g_1, g_2, g_3 are determined in a similar manner. From this it follows that under the boundary conditions of (1.18) function g has the form

$$\mu g = \pm ik\beta \pm (\alpha-\alpha_j)\left[k^{4/3}\frac{1-i}{\sqrt{2}}\left(\frac{\lambda}{\rho}\right)^{1/3} + k^{2/3}\frac{1+i}{\sqrt{2}}\left(\frac{\rho}{\lambda}\right)^{1/3}\right] \pm \pm\frac{(\alpha-\alpha_j)^2}{2}\left(ik\frac{\rho'}{\rho} + k^{1/3}\frac{9\rho'}{\lambda}\right) + \ldots \tag{2.16}$$

Since k is a sufficiently large parameter, the sign of the real part of (2.16) is determined by the terms containing k in the highest power. For these terms the sign may be selected to be such as to produce the solution which attenuates in going further away into the interior from the considered edge.

By comparing the functions of kf and μg we note that the real part of the first function has a term $k^{2/3}$ and the second function -- the term $k^{4/3}$. This corresponds to the fact that the edge effect is attenuated more rapidly than the principal stressed state.

3. Let us consider the case when $\gamma = 1$. From (1.14) and (1.17) it follows that functions f and g will have the form /110

$$kf = kf_0 + k^{1/3}f_1, \; \mu g = k^{3/2}\left(g_0 + k^{-1/2}g_1 + k^{-1}g_2\right) \tag{3.1}$$

Keeping in mind the solutions for these functions, obtained in section 1 at boundary conditions (1.16) and (1.18), we obtain formulae

$$kf = \pm ik\beta \pm k^{1/2}\frac{1+i}{\sqrt{2}}(\alpha-\alpha_j) \tag{3.2}$$

$$k^{3/2}g = \pm ik\beta \pm (\alpha-\alpha_j)\left(k^{3/2}\frac{1-i}{\sqrt{2}}\sqrt{\frac{\lambda}{\rho}} + k^{1/2}\frac{1+i}{\sqrt{2}}\sqrt{\frac{\rho}{\lambda}}\right) \pm \pm ik\frac{\rho'}{2\rho}(\alpha-\alpha_j)^2 + \ldots \tag{3.3}$$

Since for our case the function $W^{(1)}$ is given by the second formula in (1.10) when $\xi = 2$, then for the determination of $W_0^{(1)}$ we have the following differential equation

$$\frac{\partial W_0^{(1)}}{d\alpha} - \frac{\rho'}{\lambda}(\alpha-\alpha_j)W_0^{(1)} = 0 \tag{3.4}$$

The solution of this equation is

$$W_0^{(1)} = (a_0 + ib_0)\exp\frac{\rho'(\alpha-\alpha_j)^2}{2\lambda} \tag{3.5}$$

Here a_0 and b_0 are random functions of β.

For the following functions $W_{1/2}^{(1)}$, $W_1^{(1)}$ we have nonhomogeneous differential equations with the same homogeneous part as (3.4).

From (3.2) and (3.5) it follows that the principal stressed state of a cylinder is attenuated in any direction if the following condition is fulfilled

$$l \ll k^{1/2}\left|\frac{\lambda\sqrt{2\lambda\rho}}{\rho'}\right| \tag{3.6}$$

In the first approximation let us consider the boundary conditions, which must be imposed on the functions $W^{(1)}$ and $W^{(2)}$.

Function W from (1.9), taking into account (3.2) - (3.4), may be written as follows:

$$\begin{aligned} W = e^{ik\beta}\{&(a_1+ib_2)\exp r_1(\alpha-\alpha_1)+(c_1+id_1)\exp[r_3(\alpha-\alpha_1)+r(\alpha-\alpha_1)^2]+\\ &+(a_2+ib_2)\exp r_2(\alpha-\alpha_2)+(c_2+id_2)\exp[r_4(\alpha-\alpha_2)+r(\alpha-\alpha_2)^2]\}+\\ +e^{-ik\beta}\{&[(a_3+ib_3)\exp r_1(\alpha-\alpha_1)+(c_3+id_3)\exp[r_3(\alpha-\alpha_1)\,r(\alpha-\alpha_1)^2]+\\ &+(a_4+ib_4)\exp r_2(\alpha-\alpha_2)+(c_4+id_4)\exp[r_4(\alpha-\alpha_2)-r(\alpha-\alpha_2)^2]\} \end{aligned} \tag{3.7}$$

The boundaries of the cylinder are assigned by the equations $\alpha = \alpha_1$ and $\alpha = \alpha_2$ (it is considered that $\alpha_1 < \alpha_2$), and functions r_i ($i = 1, 2, 3, 4$) and r have the form

$$\begin{gathered} r_{1,2} = \mp k^{1/2}\frac{1+i}{\sqrt{2}}\left(\frac{\rho}{\lambda}\right)^{1/2}, \quad r = ik\frac{\rho'}{\rho}\\ r_{3,4} = \mp\left[k^{1/2}\frac{1-i}{\sqrt{2}}\left(\frac{\lambda}{\rho}\right)^{1/2}+k^{1/2}\frac{1+i}{\sqrt{2}}\left(\frac{\rho}{\lambda}\right)^{1/2}\right] \end{gathered} \tag{3.8}$$

The sign of these functions was selected to be such that in (3.7) the solution would attenuate towards the interior of the shell. In (3.7) the quantities a_i, b_i, c_i, d_i ($i = 1, 2, 3, 4$) are random functions of the form

$$P = kj\,(P_0 + k^{-1/2}P_1 + \ldots + k^{-1/2r}P_r)] \tag{3.9}$$

Here $P_0, P_1, \ldots, P_{r-1}$ are functions of α and β, independent of parameter k. The problem lies in formulating boundary conditions for P_0.

Taking $2Ehw = w°$, we find that

$$w° = \frac{k^3}{\lambda}\operatorname{Re} W, \quad c = \operatorname{Im} W \tag{3.10}$$

The formulas for function $u_1^{(0)} = 2Ehu_1$, $v_1^{(0)} = 2Ehv_1$, $\gamma_1° = 2Eh\gamma_1$, T_1, S_1, N_1, G_1 expressed in terms of w° and c are taken from [4].

For the cylindrical shell they have the form /111

$$u_1^\circ = \mathrm{Im}\left(-\frac{\sigma}{\lambda}\frac{\partial W}{\partial \alpha} + \frac{1}{\lambda}\int \frac{\partial^2 W}{\partial \beta^2}\, d\alpha\right) \tag{3.11}$$

$$v_1^\circ = \frac{k^3}{\lambda}\,\mathrm{Re}\int \frac{W}{\rho}\, d\beta + \mathrm{Im}\left(\frac{1}{\lambda}\int \frac{d^2 W}{d\alpha^2}\, d\beta - \sigma \frac{\partial W}{\partial \beta}\right), \quad \gamma_1^\circ = -\frac{k^3}{\lambda}\,\mathrm{Re}\,\frac{\partial W}{\partial \alpha}$$

$$T_1 = \frac{1}{\lambda^2}\,\mathrm{Im}\,\frac{\partial^2 W}{\partial \beta^2}, \quad N_1 = -\frac{k^{-3}}{\lambda}\,\mathrm{Re}\left(\frac{\partial^3 W}{\partial \alpha^3} + \frac{\partial^3 W}{\partial \alpha\, \partial \beta^2}\right)$$

$$S_1 = -\frac{1}{\lambda^2}\,\mathrm{Im}\,\frac{\partial^2 W}{\partial \alpha\, \partial \beta}, \quad G_1 = -\frac{k^{-3}}{\lambda}\,\mathrm{Re}\left(\frac{\partial^2 W}{\partial \alpha^2} + \sigma \frac{\partial^2 W}{\partial \beta^2}\right)$$

Analogous formulas may be written for the remaining stresses, moments and displacements. The formulas (3.11) and formulas similar to them enable determination of the stressed state of the shell in the first and second approximation, if the boundary conditions for the factoring coefficients in (3.9) are known.

Let us determine the shape of boundary conditions in the first approximation at an edge of the shell $\alpha = \alpha_1$ (at the $\alpha = \alpha_2$ boundary the conditions are analogous). If in (3.7) we consider that $\alpha = \alpha_1$, then the terms containing the exponent $\alpha_2 - \alpha_1$, will be small, and they may be neglected. Then (3.7) will acquire the form

$$W = e^{ik\beta}\{(a_1 + ib_2)\exp r_1(\alpha - \alpha_1) + (c_1 + id_1)\exp[r_3(\alpha - \alpha_1) + r(\alpha - \alpha_1)^2]\} + \\ + e^{-ik\beta}\{(a_3 + ib_3)\exp r_1(\alpha - \alpha_1) + (c_3 + id_3)\exp[r_3(\alpha - \alpha_1) - r(\alpha - \alpha_1)^2]\} \tag{3.12}$$

Using the Euler formulas, the integral of equation (1.5) may be separated into two parts, containing factors cos $k\beta$ and sin $k\beta$, respectively.

Taking into account (3.9), we have

$$a_1 + a_3 = k^l A^{(1)},\ a_1 - a_3 = k^{l_1} A^{(2)},\ b_1 + b_3 = k^m B^{(1)},\ b_1 - b_3 = k^{m_1} B^{(2)}$$
$$c_1 + c_3 = k^p C^{(1)},\ c_1 - c_3 = k^p C^{(2)},\ d_1 + d_3 = k^q D^{(1)},\ d_1 - d_3 = k^q D^{(2)}$$

and considering that $A^{(j)}$, $B^{(j)}$, $C^{(j)}$ and $D^{(j)}$ are expanded into series in the form (3.9).

It is necessary to find the value for function $A^{(j)}$, $B^{(j)}$, $C^{(j)}$ and $D^{(j)}$ when $\alpha = \alpha_1$ in the first approximation. For this purpose the exponents ℓ, ℓ_1, m, m_1, p, q are determined in such a manner that equations with $A_0^{(j)}$, $B_0^{(j)}$, $C_0^{(j)}$ and $D_0^{(j)}$ would have solutions (it is possible to show that the solutions will exist for equations which determine the same quantities with lower exponents 1, 2...). Let us calculate in the first approximation the functions u_1°, v_1°, w_1° and γ_1°, T_1, S_1, N_1, G_1 at the boundary $\alpha = \alpha_1$ with the use of formulae (3.11) - (3.13).

$$u_1^\circ = \frac{1}{\lambda}\left[\left(\frac{\lambda}{2\rho}\right)^{1/2}(-k^{l+3/2}A_0^{(1)} + k^{m+3/2}B_0^{(1)}) + \sigma\left(\frac{\lambda}{2\rho}\right)^{1/2}(k^{q+3/2}D_0^{(1)} - k^{p+3/2}C_0^{(1)})\right]\cos k\beta \tag{3.13}$$

$$v_1^\circ = \left(k^{l+2}\frac{1}{\rho}A_0^{(1)} + k^{m+1}\frac{\sigma}{\lambda}B_0^{(1)}\right)\sin k\beta, \qquad w_1^\circ = \frac{1}{\lambda}(k^{l+3}A_0^{(1)} + k^{p+3}C_0^{(1)})\cos k\beta$$

$$\gamma_1^{(0)} = \frac{k^3}{\lambda}\left[\left(\frac{\rho}{2\lambda}\right)^{1/2}(k^{l+1/2}A_0^{(1)} - k^{m+1/2}B_0^{(1)}) + \left(\frac{\lambda}{2\rho}\right)^{1/2}(lk^{p+1/2}C_0^{(1)} + k^{q+1/2}D_0^{(1)})\right]\cos k\beta$$

$$T_1 = -\frac{1}{\lambda^2}\left(k^{m+2}B_0^{(1)} + k^{q+2}D_0^{(1)}\right)\cos k\beta \tag{3.14}$$

$$S_1 = -\frac{1}{\lambda_2}\left[\left(\frac{\rho}{2\lambda}\right)^{1/2}\left(k^{m+3/2}B_0^{(1)} + k^{l+3/2}A_0^{(1)}\right) + \left(\frac{\lambda}{2\rho}\right)^{1/2}\left(k^{q+5/2}D_0^{(1)} - k^{p+5/2}C_0^{(0)}\right)\right]\sin k\beta$$

$$N_1 = -\frac{k^{-3}}{\rho^{3/2}\sqrt{2\lambda}}\left(k^{p+9/2}C_0^{(1)} - k^{q+9/2}D_0^{(1)}\right)\cos k\beta$$

$$G_1 = -\frac{k^{-3}}{\lambda}\left[k^{l+2}\sigma A_0^{(1)} - k^{q+3}\frac{\lambda}{\rho}D_0^{(1)} - k^{p+2}(2-\sigma)C_0^{(1)}\right]\cos k\beta \tag{3.15}$$

Here $A_0^{(1)}$, $B_0^{(1)}$, $C_0^{(1)}$, $D_0^{(1)}$ designate the boundary values of these quantities. In addition, we take into account the fact that $A_0^{(2)}$, $B_0^{(2)}$, $C_0^{(2)}$, $D_0^{(2)}$ must equal zero, in order for the factors sin kβ and cos kβ at u_1°, v_1°, w_1°, γ_1° in equation (3.14) to be the same as in the case of boundary conditions (1.7).

In subsequent notations we omit factors cos kβ and sin kβ. /112

The quantities ℓ, m, p, q, selected according to formulae (3.14) at boundary conditions of (1.7) are given in Table 1.

TABLE 1

	$u_1^\circ = 1$	$v_1^\circ = 1$	$w_1^\circ = 1$	$\gamma_1^\circ = 1$
l	$-3/2$	-2	-3	$-11/2$
m	$-5/2$	-2	-2	$-9/2$
p	$-5/2$	-2	-2	$-11/2$
q	$-5/2$	-2	-2	$-9/2$

Making use of Table 1, the boundary conditions (1.7) and (3.14) we obtain boundary values of $A_0^{(1)}$, $B_0^{(1)}$, $C_0^{(1)}$ and $D_0^{(1)}$ and then by means of (3.15) and Table 1 we obtain, as in [5], Table 2 for elastic reactions. This table is symmetrical, as it must be according to the reciprocity principle.

TABLE 2

	$u_1^\circ = 1$	$v_1^\circ = 1$	$w_1^\circ = 1$	$\lambda\gamma_1^\circ = 1$
λT_1	$-k^{3/2}(2\rho/\lambda)^{1/2}$	$-2(1-\sigma)\rho/\lambda$	$k^{-1}(1-2\sigma)$	$-k^{-5/2}(1-\sigma)(2\rho/\lambda)^{1/2}$
λS_1	$-2(1-\sigma)\rho/\lambda$	$-k^{-1/2}(2\rho/\lambda)^{1/2}$	$k^{-1/2}(2\lambda/\rho)^{1/2}$	$-k^2$
λN_1	$k^{-1}(1-2\sigma)$	$k^{-1/2}(2\lambda/\rho)^{1/2}$	$-k^{3/2}(\lambda/\rho)^{-3/2}\sqrt{2}$	$k^{-3}\lambda/\rho$
G_1	$-k^{-5/2}(1-\sigma)(2\rho/\lambda)^{1/2}$	$-k^2$	$k^{-3}\lambda/\rho$	$-k^{-9/2}(2\lambda/\rho)^{1/2}$

4. Finally, let us consider the case when γ = 2. From (1.14) and (1.17) it follows that functions f and g have the following form

$$kf = kf_0, \quad \mu g = k^2 g_0 + kg_1 \tag{4.1}$$

The values for these functions under boundary conditions (1.16) and (1.18) will be

$$f = \pm i\beta, \quad k^2 g = \pm ik\beta \pm k^2\frac{1-i}{\sqrt{2}}\left(\frac{\lambda}{\rho}\right)^{1/2}(\alpha - \alpha_j) \pm ik\frac{\rho'}{2\rho}(\alpha-\alpha_j)^2 + \dots \tag{4.2}$$

Here the function f is purely imaginary. This means that when $\gamma = 2$ the exponential attenuation (with a large factor in the exponent) of the principal stressed state along the generatrices of the cylinder is absent.

Let us turn to construct function $W_i^{(1)}$ from (1.12). In the considered case the factor ξ must equal 1. Then for the determination of $W_0^{(1)}$ an equation is obtained

$$L_2 W_0^{(1)} + N_0 W_0^{(1)} = 0 \tag{4.3}$$

It includes the operator N_0. This means that function $W_0^{(1)}$ is determined taking into account the moment operator N. By substituting the value for the operators L_2 and N_0 into (4.3), we obtain a differential equation for W_0

$$\frac{\partial^2 W_0^{(1)}}{\partial \alpha^2} - i\frac{\rho}{\lambda} W_0^{(1)} = 0 \tag{4.4}$$

For the function $W_0^{(1)}$ the second order equation was obtained. This occurs because the lines β = const are double characteristics of the momentless operator.

The solution of equation (4.4) has the form

$$W_0^{(1)} = A_0 \exp\left[\frac{1+i}{\sqrt{2}}\left(\frac{\rho}{\lambda}\right)^{1/2}(\alpha-\alpha_j)\right] + B_0 \exp\left[-\frac{1+i}{\sqrt{2}}\left(\frac{\rho}{\lambda}\right)^{1/2}(\alpha-\alpha_j)\right] \quad (j = 1, 2) \tag{4.5}$$

Here A_0 and B_0 are unknown functions of β.

For the remaining $W_0^{(1)}$ ($l = 2, 3, \ldots, r - 1$) functions we also have equation (4.4) but it is nonhomogeneous.

The solution for $W_\ell^{(2)}$ in the integral, corresponding to the edge effect, can be obtained in the form of the following series with powers of $\alpha - \alpha_j$:

$$W_0^{(2)} = C_0 + iD_0 \pm \frac{1+i}{\sqrt{2}}\left(\frac{\rho}{\lambda}\right)^{1/2}(C_0 + iD_0)(\alpha-\alpha_j) + \ldots \tag{4.6}$$

Here C_0 and D_0 are unknown functions of β.

Using formulas (4.2) and (4.5) we obtain the following solutions for the equation (1.5) /113

$$\begin{aligned} W = \Big\{&(a_1 + ib_1)\exp\left[-\frac{1+i}{\sqrt{2}}\left(\frac{\rho}{\lambda}\right)^{1/2}(\alpha-\alpha_1)\right] + (a_2 + ib_2)\exp\left[\frac{1+i}{\sqrt{2}}\left(\frac{\rho}{\lambda}\right)^{1/2}(\alpha-\alpha_2)\right] + \\ &+ (c_1 + id_1)\exp\left[-k^2\frac{1-i}{\sqrt{2}}\left(\frac{\lambda}{\rho}\right)^{1/2}(\alpha-\alpha_1) + ik\frac{\rho'}{2\rho}(\alpha-\alpha_1)^2\right] + \\ &+ (c_2 + id_2)\exp\left[k^2\frac{1-i}{\sqrt{2}}\left(\frac{\lambda}{\rho}\right)^{1/2}(\alpha-\alpha_2) + ik\frac{\rho'}{2\rho}(\alpha-\alpha_2)^2\right]\Big\} e^{ik\beta} + \\ &+ \Big\{(a_3 + ib_3)\exp\left[-\frac{1+i}{\sqrt{2}}\left(\frac{\rho}{\lambda}\right)^{1/2}(\alpha-\alpha_1)\right] + (a_4 + ib_4)\exp\left[\frac{1+i}{\sqrt{2}}\left(\frac{\rho}{\lambda}\right)^{1/2}(\alpha-\alpha_2)\right] + \end{aligned} \tag{4.7}$$

$$+(c_3+id_3)\exp\left[-k^2\frac{1-i}{\sqrt{2}}\left(\frac{\lambda}{\rho}\right)^{1/2}(\alpha-\alpha_1)-ik\frac{\rho'}{2\rho}(\alpha-\alpha_1)^2\right]+$$

$$+(c_4+id_4)\exp\left[k^2\frac{1-i}{\sqrt{2}}\left(\frac{\lambda}{\rho}\right)^{1/2}(\alpha-\alpha_2)-ik\frac{\rho'}{2\rho}(\alpha-\alpha_2)^2\right]\Bigg\}e^{-ik\beta} \qquad (4.7) \text{ Cont'd}$$

Here the terms, containing as their cofactor exponential functions with large factor k^2 in the exponent, are the integrals of the edge effect. The remaining terms are the integrals of the principal stressed state.

During formulation of the boundary conditions when $\alpha = \alpha_1$ ($\alpha = \alpha_2$) in (4.7) one should neglect those terms which contain functions c_2, d_2, c_4, d_4 (c_1, d_1, c_3, d_3) since they are sufficiently small. For a long shell, in addition to this it is necessary to disregard those terms which contain functions a_2, b_2, a_4, b_4 (a_1, b_1, a_3, b_3) when $\alpha = \alpha_1$ ($\alpha = \alpha_2$).

Let us consider the case when a shell is short. We shall replace the functions a_j, b_j, c_j, d_j by $A^{(j)}$, $B^{(j)}$, $C^{(j)}$, $D^{(j)}$ ($j = 1, 2, 3, 4$) from the formulae analogous to those in (3.13). These functions have comparable values at both edges of the shell.

Thus, the problem is reduced to the determination of sixteen functions of β.

Let us express the displacement u_1^0, v_1^0, w_1^0 and the angle of rotation γ_1^0 through the unknown functions $A_0^{(j)}$, $B_0^{(j)}$, $C_0^{(j)}$, $D_0^{(j)}$ ($j = 1, 2, 3, 4$). For the determination of these functions let us use the conditions of (1.7) at the boundaries $\alpha = \alpha_1$ and $\alpha = \alpha_2$. We obtain eight equations, each of which may be represented as

$$P_j\cos k\beta + Q_j\sin k\beta = 0 \qquad (j = 1, 2, \ldots, 8)$$

where P_j and Q_j are only the functions of β, and cos $k\beta$ and sin $k\beta$ depend also on the parameter k. Let us here set $P_j = 0 \ldots$, $Q_j = 0$ ($j = 1,8$). Thus, sixteen equations are obtained with sixteen unknowns, which in contrast to those in paragraph 3 are not separated into two independent systems. It is too cumbersome to construct the sought functions in the general form: it is much simpler to calculate these functions at individual points.

For a long shell the fulfillment of the boundary conditions is much easier. Let us calculate $A_0^{(j)}$, $B_0^{(j)}$, $C_0^{(j)}$, $D_0^{(j)}$ at the boundary $\alpha = \alpha_1$. In the first approximation the function W, keeping in mind (4.7) and the symmetrical nature of the boundary conditions (1.7), may be written as

$$W = \left\{(k^l A_0^{(1)} + ik^m B_0^{(1)})\exp\left[-\frac{1+i}{\sqrt{2}}\left(\frac{\rho}{\lambda}\right)^{1/2}(\alpha-\alpha_1)\right]+\right.$$

$$\left.+(k^p C_0^{(1)} + ik^q D_0^{(1)})\exp\left[-k^2\frac{1-i}{\sqrt{2}}\left(\frac{\lambda}{\rho}\right)^{1/2}(\alpha-\alpha_1)\right]\right\}\cos k\beta \qquad (4.8)$$

Substituting (4.8) into (3.10) and (3.11) at the $\alpha = \alpha_1$ boundary we obtain:

$$
\begin{gathered}
u_1^\circ = \frac{k^2}{\lambda}\left(\frac{\lambda}{2\rho}\right)^{1/2}\left[-k^l A_0^{(1)} + k^m B_0^{(1)} + \sigma\left(-k^p C_0^{(1)} + k^q D_0^{(1)}\right)\right]\cos k\beta \\
v_1^\circ = \left[\frac{k^{l+3}}{\rho} A_0^{(1)} + \frac{\sigma}{\lambda}\left(k^{m+1} B_0^{(1)} + k^{q+1} D_0^{(1)}\right)\right]\sin k\beta \\
w_1^\circ = \frac{k^4}{\lambda}\left(k^l A_0^{(1)} + k^p C_0^{(1)}\right)\cos k\beta \qquad (4.9) \\
\gamma_1^\circ = \frac{1}{\lambda^2}\left[\left(\frac{\lambda}{2\rho}\right)^{1/2}\left(k^{p+6} C_0^{(1)} + k^{q+6} D_0^{(1)}\right) + \left(\frac{\rho}{2\lambda}\right)^{1/2}\left(k^{l+4} A_0^{(1)} - k^{m+4} B_0^{(1)}\right)\right]\cos k\beta \\
T_1 = -\frac{k^2}{\lambda^2}\left(k^m B_0^{(1)} + k^q D_0^{(1)}\right)\cos k\beta \\
S_1 = \frac{k}{\lambda^2}\left[\left(\frac{\rho}{2\lambda}\right)^{1/2}\left(-k^l A_0^{(1)} - k^m B_0^{(1)}\right) + \left(\frac{\lambda}{2\rho}\right)^{1/2}\left(k^{p+2} C_0^{(1)} - k^{q+2} D_0^{(1)}\right)\right]\sin k\beta \\
N_1 = \frac{1}{\rho\sqrt{2\lambda\beta}}\left(k^{q+2} D_0^{(1)} - k^{p+2} C_0^{(1)}\right)\cos k\beta \\
G_1 = \frac{1}{\lambda}\left[-k^q \frac{\lambda}{\rho} D_0^{(1)} + \sigma\left(k^{l-2} A_0^{(1)} + k^{p-2} C_0^{(1)}\right)\right]\cos k\beta
\end{gathered}
$$

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TABLE 3

	$u_1^\circ = 1$	$v_1^\circ = 1$	$w_1^\circ = 1$	$\gamma_1^\circ = 1$
l	−4	−3	−6	−8
m	−2	−3	−4	−6
p	−4	−3	−4	−8
q	−4	−3	−4	−6

When the boundary conditions of (1.7) are fulfilled the exponents l, m, p, q have the values which are given in Table 3.

Table 4 gives the values for stresses T_1, S_1, N_1 and G_1 when $\alpha = \alpha_1$.

TABLE 4

	$u_1^\circ = 1$	$v_1^\circ = 1$	$w_1^\circ = 1$	$\lambda\gamma_1^\circ = 1$
λT_1	$-(2\rho/\lambda)^{1/2}$	$-k^{-1}2(1-\sigma)\rho/\lambda$	$k^{-2}(1-2\sigma)$	$-k^{-4}(1-\sigma)(2\rho/\lambda)^{1/2}$
λS_1	$-k^{-1}2(1-\sigma)\rho/\lambda$	$-(2\rho/\lambda)^{1/2}$	$k^{-1}(2\lambda/\rho)^{1/2}$	$-k^{-3}$
λN_1	$k^{-2}(1-\sigma)$	$k^{-1}(2\lambda/\rho)^{1/2}$	$-k^{-2}(\lambda/\rho)^{3/2}\sqrt{2}$	$k^{-4}\lambda/\rho$
G_1	$-k^{-4}(1-\sigma)(2\rho/\lambda)^{1/2}$	$-k^{-3}$	$k^{-4}\lambda/\rho$	$-k^{-6}(2\lambda/\rho)^{1/2}$

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